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Higher order addition laws on Abelian varieties and the fractional quantum Hall effect

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Abstract

Addition formulas for theta functions of arbitrary order are shown and applied to the theoretical understanding of the fractional quantum Hall effect in a multi-layer two-dimensional many-electron system under periodic conditions.

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Subj. Class: Differential geometry; Quantum mechanics

1. Introduction

As is well known, classical addition formulas for theta functions are formulas of degree 2. In this paper we prove addition formulas of arbitrary degree for theta functions. The explicit generalized addition formulas are stated in Theorem 2.6 and Proposition 2.7 and the main ingredients in the proof are the cube theorem and the isogeny theorem of Mumford [1,3].

We apply these results to offer a formulation of the fractional quantum Hall effect in a multi-layer many-electron system and possible generalizations. In the study of the ordinary

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quantum Hall effect under periodic conditions [4], the basic geometric object is an algebraic torus

$$\Sigma = \frac{\mathbf{C}}{\mathbf{Z} + \tau\mathbf{Z}},$$

and study of the isogeny

$$\varphi_N : \Sigma^N \to \Sigma^N, \qquad \varphi(x_1, \ldots, x_N) = (x_1 + \cdots + x_N, x_1 - x_2, \ldots, x_1 - x_N)$$

determines the change of coordinates from one-particle to center-of-mass and relative coordinates in the space of wave functions of the system. The change from one-particle coordinates, $(x_1, \ldots, x_N)$, to center-of-mass and relative coordinates, $(x_1 + \cdots + x_N, x_1 - x_2, \ldots, x_1 - x_N)$ is crucial in the definition of the Laughlin wave function describing the ground state of the FQHE on the complex plane. On the algebraic torus, Σ, this change of coordinates becomes the isogeny mentioned above. In the second quantization formalism, to extend the Laughlin variational principle for the fractional quantum Hall effect to the periodic case thus requires the use of generalized addition formulas for elliptic theta functions, see [4].

The goal of this work is to generalize this construction to an arbitrary algebraic torus of dimension g $X = \mathbf{C}/(\mathbf{Z}^g + \tau\mathbf{Z}^g)$ (where $\tau = \tau_{ij}$ is a complex $g \times g$ symmetric matrix with the imaginary part positive definite). From a physical point of view, this means that we are considering a system of many electrons moving on g two-dimensional layers with different periodic conditions defined by the period matrix τ_{ij}. As in our earlier paper [4], we are naturally lead to study the isogenies

$$\varphi_N : X^N \to X^N, \qquad \varphi(x_1, \ldots, x_N) = (x_1 + \cdots + x_N, x_1 - x_2, \ldots, x_1 - x_N).$$

The behavior of the global sections of some line bundles over X^N under the isogeny φ_N will determine the vector space of wave functions of the system of electrons under the generalized periodic conditions. This behavior and the link between wave functions depending either on $(x_1, \ldots, x_N)$ or $(x_1 + \cdots + x_N, x_1 - x_2, \ldots, x_1 - x_N)$ is completely described by the generalized addition formulas. We thus obtain a complete description of the space of wave functions of the FQH effect in a multi-layer two-dimensional electron system. In fact, in the real physical situation it suffices to consider a diagonal period matrix τ, unless tunnel effects between different layers are taken into account, see [13].

Having done this, we also consider the Fourier–Mukai transform of some line bundles over X^N determined by the generalized Haldane–Rezayi wave functions and the semi-stability of these transforms. We show how the slope of these bundles is related to the Hall conductivity, which therefore appears as a topological invariant. There is an isomorphism between the Fourier–Mukai transforms for any number of electrons N and, thus, the Hall conductivity depends only on the center-of-mass dynamics characterized by the Haldane–Rezayi states.

The organization of the paper is as follows. In Section 2, we show the main theorems and establish the generalized addition formulas for Abelian varieties. In Section 3, the vector spaces for the quantum ground states are constructed in terms of higher order odd theta functions. Section 4 is devoted to studying the Fourier–Mukai transform of the line bundles over X^N related to the quantum vector spaces. In Section 5, all these developments are applied to the analysis of the fractional quantum Hall effect in multi-layer periodic electron systems. Finally, in Section 6 a comparison with the physics literature is offered and some obscure points are clarified.

2. Generalized addition formulas for Abelian varieties

Let X be an Abelian variety of dimension g over the field $\mathbb{C}$ of complex numbers. Let us define the following family of morphisms:

$$M, m_{ij}, s_{ij} : X \times \overset{N}{\cdots} \times X \to X, \qquad M(x_1, \dots, x_N) = x_1 + \cdots + x_N,$$
$$m_{ij}(x_1, \dots, x_N) = x_i + x_j, \qquad s_{ij}(x_1, \dots, x_N) = x_i - x_j,$$
$$p_i : X \times \cdots \times X \to X$$

will be the natural projections.

Theorem 2.1 (Generalized cube theorem). *For any symmetric invertible sheaf L on X one has a natural isomorphism*:

$$M^* L \simeq \left(\bigotimes_{i<j} m_{ij}^* L \right) \otimes \left(\bigotimes_{i=1}^{N} p_i^* L^{\otimes -N+2} \right).$$

Proof. This follows from the cube theorem [1], and induction over N. □

Corollary 2.2. *For any symmetric invertible sheaf L over X, one has a natural isomorphism*:

$$M^* L \otimes \left(\bigotimes_{i<j} s_{ij}^* L \right) \simeq p_1^* L^{\otimes N} \otimes \cdots \otimes p_N^* L^{\otimes N}.$$

Proof. By Theorem 2.1 one has

$$M^* L \otimes \left(\bigotimes_{i<j} s_{ij}^* L \right) \simeq \bigotimes_{i<j} \left(m_{ij}^* L \otimes s_{ij}^* L \right) \otimes \left(\bigotimes_{i=1}^{N} p_i^* L^{\otimes -N+2} \right).$$

Let us denote by $p_{ij} : X \times \cdots^{N} \times X \to X \times X$ the projection on the factors (i, j) and by $\pi : X \times X \to X$ $(i = 1, 2)$ the natural projections. One has

$$m_{ij}^* L \otimes s_{ij}^* L \simeq p_{ij}^* (\xi^* (\pi_1^* L \otimes \pi_2^* L)) \simeq p_{ij}^* (\pi_1^* L^{\otimes 2} \otimes \pi_2^* L^{\otimes 2}),$$

where $\xi : X \times X \to X \times X$ is the morphism: $\xi(x, y) = (x + y, x - y)$.

We therefore have

$$\begin{aligned} M^* L \otimes \left(\bigotimes_{i<j} s_{ij}^* L \right) &\simeq \bigotimes_{i<j} p_{ij}^* (\pi_1^* L^{\otimes 2} \otimes \pi_2^* L^{\otimes 2}) \otimes \left(\bigotimes_{i=1}^{N} p_i^* L^{\otimes -N+2} \right) \\ &\simeq \bigotimes_{i<j} (p_i^* L^{\otimes 2} \otimes p_j^* L^{\otimes 2}) \otimes \left(\bigotimes_{i=1}^{N} p_i^* L^{\otimes -N+2} \right) \\ &\simeq \bigotimes_{i=1}^{N} p_i^* L^{\otimes N}. \end{aligned}$$

□

Let us consider the morphism of Abelian varieties:

$$\xi_N : X \times \overset{N}{\cdots} \times X \to X \times \overset{r}{\cdots} \times X \left(r = \frac{N(N-1)}{2} + 1 \right) (x_1, \dots, x_N)$$
$$\to (x_1 + \cdots + x_N, x_1 - x_2, \dots, x_{N-1} - x_N).$$

By Corollary 2.2 one has an isomorphism

$$\xi_N^*(p_1^* L \otimes \cdots \otimes p_r^* L) \simeq M^* L \otimes \left(\bigotimes_{i<j} s_{ij}^* L \right) \simeq p_1^* L^{\otimes N} \otimes \cdots \otimes p_N^* L^{\otimes N},$$

which induces a homomorphism between the vector spaces of global sections:

$$\xi_N^* : H^0(X, L) \otimes \overset{r}{\cdots} \otimes H^0(X, L) \to H^0(X, L^{\otimes N}) \otimes \overset{r}{\cdots} \otimes H^0(X, L^{\otimes N}).$$

For applications to the study of the quantum Hall effect under periodic conditions, it is very important to compute explicitly the homomorphism ξ_N^* (see [4] and Section 6 of this paper).

Observe that the kernel of ξ_N is $\Delta(X_N)$, where X_N is the N-torsion subgroup of X and $\Delta : X \hookrightarrow X \times \cdots^N \times X$ is the diagonal immersion.

The morphism ξ_N factors as follows:

$$Z = X \times \overset{N}{\cdots} \times X \overset{\phi_N}{\to} Y = Z/\Delta(X_N) \overset{i}{\hookrightarrow} X \times \overset{r}{\cdots} \times X, \qquad \xi_N = i \circ \phi_N.$$

Let us set $\mathcal{L} = (p_1^* L \otimes \cdots^r \otimes p_r^* L)$.

One has that $\phi_N^* \mathcal{L}_{|Y} = M^* L \otimes \left(\otimes_{i<j} s_{ij}^* L \right) = \mathcal{M}_N$.

We can now consider the morphism

$$\varphi_N : Z \to Z, \qquad \varphi(x_1, \dots, x_N) = (x_1 + \cdots + x_N, x_1 - x_2, \dots, x_1 - x_N),$$

and define an invertible sheaf $\mathcal{R}_N$ on Z by

$$\mathcal{R}_N = (p_1^* L \otimes \cdots \otimes p_N^* L) \otimes (\otimes s_{ij}^* L).$$

One has a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\xi_N"] \arrow[dr, "\varphi_N"'] & X \times \overset{r}{\cdots} \times X \arrow[d, "\pi_{1\ldots N}"] \\ & Z \end{tikzcd}$$

$\pi_{1,\dots,N}$ being the projection on the N first factors. $\pi_{1,\dots,N}$ induces an isomorphism $Y \overset{\sim}{\to} Z$ such that

$$\pi_{1,\dots,N}^* \mathcal{R}_N \simeq \mathcal{L} \quad \text{and} \quad \xi_N^* \mathcal{L} = \varphi_N^* \mathcal{R}_N \simeq \mathcal{M}_N.$$

But φ_N is an isogeny of kernel $\Delta(X_N)$ and the problem of computing the homomorphism ξ_N^* is reduced to computing the homomorphism

$$\varphi_N^* : H^0(Z, \mathcal{R}_N) \to H^0(Z, \mathcal{M}_N).$$

To compute this homomorphism, we can apply the Mumford theory of algebraic theta functions [2].

To make explicit computations, let us fix a principal polarization (p.p.) Θ on the Abelian variety X, and assume that $L = \mathcal{O}_X(m\Theta)$; then, one has that $\mathcal{M} \simeq \otimes_{i=1}^{N} p_i^* \mathcal{O}_X(Nm\Theta)$.

For any invertible sheaf $\mathcal{F}$ on X, let us denote by $K(\mathcal{F})$ the subgroup of X which leaves $\mathcal{F}$ invariant under translations ($K(\mathcal{F}) = x \in X : T_x^* \mathcal{F} \simeq \mathcal{F}$) and by $\mathcal{G}(\mathcal{F})$ the theta-group of $\mathcal{F}$.

In our case, one has

$$K(L) = X_m = \text{subgroup of } m\text{-torsion points of } X,$$
$$K(L^{\otimes N}) = X_{Nm} \quad \text{and} \quad K(L) = N \cdot K(L^{\otimes N}) \subset X_{Nm}.$$

The isomorphism $\varphi^* \mathcal{R}_N \simeq \mathcal{M}_N$ implies that

$$\begin{aligned} K(\mathcal{M}_N) &= K(L^{\otimes N}) \times \overset{N}{\cdots} \times K(L^{\otimes N}) \\ &= X_{Nm} \times \overset{N}{\cdots} \times X_{Nm} \supset K(\mathcal{R}_N) \supset X_m \times \overset{N}{\cdots} \times X_m. \end{aligned}$$

For any invertible sheaf $L = \mathcal{O}_X(D)$ on an Abelian variety X of dimension g, let us denote by $\deg(L)$ the number D^g.

Proposition 2.3.

$$|K(\mathcal{R}_N)| = N^{2(N-2)g} m^{2Ng}, \qquad \deg \mathcal{R}_N = (N_g)_! N^{(N-2)g} m^{Ng}.$$

Proof. Observe that $\ker \varphi_N = \Delta(X_N) \simeq X_N$. One then has that

$$\deg \varphi_N^* \mathcal{R}_N = \deg \varphi_N^* \deg \mathcal{R}_N = N^{2g} \mathcal{R}_N,$$

and

$$\deg \varphi_N^* \mathcal{R}_N = (N_g)_! N^{Ng} m^{Ng}.$$

Therefore: $\deg \mathcal{R}_N = (N_g)_! N^{(N-2)g} m^{Ng}$. □

The structure of the group $K(\mathcal{R}_N)$ is given by the following theorem.

Theorem 2.4. *$K(\mathcal{R}_N)$ is the subgroup of points $\varphi_N(p) = (x_1 + \cdots + x_N, x_1 - x_2, \ldots, x_1 - x_N) \in X \times \cdots^N \times X$ such that: $p = (x_1, \ldots, x_N) \in X_{Nm} \times \cdots^N \times X_{Nm}$ and $x_1 + \cdots + x_N \in X_m$.*

In particular, $K(\mathcal{R}_N)$ has subgroups isomorphic to $X_N \times \cdots^N \times X_N$ given by

$$X_m \times \overset{N}{\cdots} \times X_m \hookrightarrow K(\mathcal{R}_N), \qquad (x_1, \ldots, x_N) \to (x_1, \ldots, x_N)$$

(*with respect to the natural immersion $X_m = N \cdot X_{Nm} \subset X_{mN}$*) *and*

$$\begin{aligned} &X_N \times \overset{N-2}{\cdots} \times X_N \hookrightarrow K(\mathcal{R}_N) \hookrightarrow X \times \overset{N}{\cdots} \times X, \\ &(x_2, \ldots, x_{N-1}) \to (0, -x_2, \ldots, -x_{N-1}, x_2 + \cdots + x_{N-1}). \end{aligned}$$

Proof. Let $\hat{X} = \mathrm{Pic}^0(X)$ be the dual Abelian variety. From the exact sequence

$$0 \to X_N \xrightarrow{\Delta} X \times \overset{N}{\cdots} \times X \xrightarrow{\varphi_N} X \times \overset{N}{\cdots} \times X \to 0,$$

one deduces the existence of the following dual exact sequence

$$0 \to \hat{X}_N \xrightarrow{\Delta} \hat{X} \times \overset{N}{\cdots} \times \hat{X} \xrightarrow{\varphi_N^*} \hat{X} \times \overset{N}{\cdots} \times \hat{X} \to 0,$$

which means that given a point $p = (x_1, \dots, x_N) \in K(\mathcal{M}_N)$, one has

$$T^*_{\phi_N(p)}\mathcal{R}_N \otimes \mathcal{R}_N^{\otimes -1} \simeq p_1^* M \otimes \cdots \otimes p_N^* M$$

for a certain invertible sheaf M of degree zero on X. By restricting this equality to $X \times \{e\} \times \cdots \{e\}$ we compute M and obtain the following isomorphism:

$$T^*_{\phi_N(p)}\mathcal{R}_N \otimes \mathcal{R}_N^{\otimes -1} \simeq p_1^*(T^*_{x_1+\cdots+x_N} L \otimes L^{\otimes -1}) \otimes \cdots \otimes p_N^*(T^*_{x_1+\cdots+x_N} L \otimes L^{\otimes -1}).$$

Then, $T^*_{\phi_N(p)}\mathcal{R}_N \simeq \mathcal{R}_N$ if and only if $x_1 + \cdots + x_N \in X_m$. The rest of the theorem follows easily from this result. □

Remark 2.5. We have constructed two subgroups, $X_m \times \cdots^N \times X_m$ and $X_N \times \cdots^{N-2} \times X_N$ of $K(\mathcal{R}_N)$. Thus if $(m, N) = 1$, a general element of $K(\mathcal{R}_N)$ has the form

$$(y_1, y_2 - x_2, \dots, y_{N-1} - x_{N-1}, y_N + x_2 + \cdots + x_{N-1}),$$

where $(y_1, \dots, y_N) \in X_m^N$ and $(x_2, \dots, x_{N-1} \in X_N^{N-2})$.

Let us fix compatible theta-structures [2] on L and $L^{\otimes N}$. These theta-structures induce compatible theta-structures on $\mathcal{R}_N$ and $\mathcal{M}_N$ and decompositions

$$\begin{aligned}
&K(L) \simeq A(L) \times B(L), \quad A(L) \simeq (\mathbb{Z}/m\mathbb{Z})^g, \qquad K(L^{\otimes N}) \simeq A(L^{\otimes N}) \times B(L^{\otimes N}),\\
&A(L^{\otimes N}) \simeq (\mathbb{Z}/mN\mathbb{Z})^g, \qquad K(\mathcal{M}_N) \simeq A(L^{\otimes N})^N \times B(L^{\otimes N})^N,\\
&K(\mathcal{R}_N) \simeq A(\mathcal{R}_N) \times B(\mathcal{R}_N),
\end{aligned}$$

where $B(\mathcal{R}_N) \subset B(L^{\otimes N})^N$, and by Theorem 2.4 one has

$$B(L)^N \subset B(\mathcal{R}_N), \qquad m \cdot B(L^{\otimes N})^{N-2} \subset B(\mathcal{R}_N), \qquad B(L)^N = N \cdot B(L^{\otimes N})^N$$

in such a way that $B(\mathcal{R}_N)$ is the subgroup of $B(L^{\otimes N})^N$ generated by $m \cdot B(L^{\otimes N})^{N-2}$ and $N \cdot B(L^{\otimes N})^N$.

We have natural isomorphisms [2]

$$\begin{aligned}
&H^0(X, L) = V_m = \{\text{functions } B(L) \to \mathbb{C}\},\\
&H^0(X, L^{\otimes N}) = V_{Nm} = \{\text{functions } B(L^{\otimes N}) \to \mathbb{C}\}.
\end{aligned}$$

For each $d \in B(L)$, let δ_d be the global section of L defined by the characteristic function of d, and for each $b \in B(L^{\otimes N})$ let δ_b be the corresponding global section of $L^{\otimes N}$.

Observe that $H^0(Z, \mathcal{R}_N)$ is a $\mathbb{C}$-vector space of dimension $N^{(N-2)g} m^{Ng}$ and $H^0(Z, \mathcal{M}_N)$ is a $\mathbb{C}$-vector space of dimension $N^{Ng} m^{Ng}$. The following result give us an explicit description of the homomorphism $\varphi_N^* : H^0(Z, \mathcal{R}_N) \to H^0(Z, \mathcal{M}_N)$.

Theorem 2.6. *Let us assume that L and $L^{\otimes N}$ have compatible theta-structures satisfying the above conditions. For each $d \in B(\mathcal{R}_N)$ one has*

$$\varphi_N^*(\delta_d) = \lambda \sum_{\substack{b \in B(\mathcal{M}_N) \\ f(b)=d}} \delta_d,$$

where $\lambda \in \mathbb{C}$ is a constant which we will assume to be equal to 1.

Proof. This follows from the isogeny theorem [1,3]. □

This result allow us to give more explicit expressions for φ_N^*.

Given $d = (d_1, \dots, d_n) \in N \cdot B(L^{\otimes N})^N = [(\mathbb{Z}/m\mathbb{Z})^g]^N \subset B(\mathcal{R}_N)$, let us denote by δ_d the element

$$\delta_d = \delta_{d_1} \otimes \cdots \otimes \delta_{d_N} \left(\bigotimes_{\substack{i>j \\ j \geq 2}} s_{ij}^* \delta_{d_i - d_j} \right) \in H^0(Z, \mathcal{R}_N),$$

and for each $h = (0, -h_2, \dots, -h_{N-1}, h_2 + \cdots + h_{N-1}) \in [(\mathbb{Z}/N\mathbb{Z})^g]^{N-2} \subset B(\mathcal{R}_N)$ we denote by δ_h the corresponding global section of $\mathcal{R}_N$.

With these notations one has the following proposition.

Proposition 2.7.

1.
$$\varphi_N^*(\delta_d) = \theta[d_1](x_1 + \cdots + x_N) \prod_{j \geq 2} \theta[d_j](x_1 - x_j) \prod_{\substack{i>j \\ j \geq 2}} \theta[d_i - d_j](x_i - x_j)$$
$$= \lambda \sum_{\substack{b_i \in B(L^{\otimes N}) \\ b_1 + \cdots + b_N = d_1 \\ b_1 - b_2 = d_2 \\ \vdots \\ b_1 - b_N = d_N}} \theta[b_1](x_1) \theta[b_2](x_2) \cdots \theta[b_N](x_N),$$

$\theta[b_i](x_i)$ being the global section of $L^{\otimes N}$ defined by δ_{b_i} (in the ith component of X^N) and $\theta[d_i](z)$ the global section of L defined by δ_{d_i}.

2. $$\begin{aligned}\varphi_N^*(\delta_h) &= \theta_h(x_1+\cdots+x_N, x_1-x_2,\ldots,x_1-x_N)\\ &= \lambda \sum_{\substack{b_i\in B(L^{\otimes N})\\ b_1+\cdots+b_N=0\\ b_1-b_2=-h_2\\ \vdots\\ b_1-b_{N-1}=-h_{N-1}\\ b_1-b_N=h_2+\cdots+h_{N-1}}} \theta[b_1](x_1)\theta[b_2](x_2)\cdots\theta[b_N](x_N)\\ &= \lambda\sum_{(b_1,\ldots,b_N)}\theta[b_1](x_1)\theta[b_2+h_2](x_2)\cdots\theta[b_{N_1}+h_{N-1}]\\ &\quad\times(x_{N-1})\theta[b_N-h_2-\cdots-h_{N-1}](x_N),\end{aligned}$$

where $(b_1,\ldots,b_N)\in\ker(B(\mathcal{M}_N))$ *and in both formulae* λ *is a constant independent of* d *and* h.

Proof. This follows easily from Theorem 2.6 and the description of $K(\mathcal{R}_N)$. □

Remark 2.8. In the case $(m,N)=1$, a general element of $B(\mathcal{R}_N)$ takes the form

$$d=(d_1,d_2-h_2,\ldots,d_{N-1}-h_{N-1},d_N+h_2+\cdots+h_{N-1}),$$

where $(d_1,\ldots,d_N)\in B(L)^N$ and $(h_2,\ldots,h_{N-1})\in[(\mathbb{Z}/N\mathbb{Z})^g]^{N-2}$, and the general addition formula is

$$\begin{aligned}\varphi_N^*(\delta_d) &= \theta_h(x_1+\cdots+x_N, x_1-x_2,\ldots,x_1-x_N)\\ &= \lambda \sum_{\substack{b_i\in B(L^{\otimes N})\\ b_1+\cdots+b_N=d_1\\ b_1-b_2=d_2-h_2\\ \vdots\\ b_1-b_{N-1}=d_{N-1}-h_{N-1}v\\ b_1-b_N=d_N+h_2+\cdots+h_{N-1}}} \theta[b_1](x_1)\theta[b_2](x_2)\cdots\theta[b_N](x_N).\end{aligned}$$

Remark 2.9. We have explicitly computed the homomorphism of vector spaces $\varphi_N^*: H^0(Z,\mathcal{R}_N)\to H^0(Z,\mathcal{M}_N)$. If we wish to compute $\xi_N^*: H^0(X^r,\mathcal{L})\to H^0(Z,\mathcal{M}_N)$, let us note that we have the commutative diagram

$$\begin{tikzcd} H^0(X^r,\mathcal{L}) \arrow[rr, "\xi^*_N"] \arrow[d, "i^*"'] & & H^0(Z,\mathcal{M}_N) \\ H^0(Y,\mathcal{L}_{|Y}) \simeq H^0(Z,\mathcal{R}_N) \arrow[urr, "\phi^*_N\equiv\varphi^*_N"'] & & \end{tikzcd}$$

and we have

$$K(\mathcal{L})\simeq X_m^r,\quad K(\mathcal{L})\cap Y\subseteq K(\mathcal{L}_{|Y})\simeq K(\mathcal{R}_N),\quad K(\mathcal{L})\cap Y\simeq X_m\times\overset{N}{\cdots}\times X_m.$$

From these identities one can easily prove that the vector subspace $\xi_N^* H^0(X^r, \mathcal{L}) \subseteq H^0(Z, \mathcal{M}_N)$ can be identified with the subspace generated by the global sections $\{\varphi_N^*(\delta_d)\}$ defined in 2.7(a).

3. Vector spaces of higher order odd theta functions

We shall apply the results of the first section to compute some vector spaces of theta functions which are relevant in the study of the fractional quantum Hall effect (for a similar discussion for elliptic curves, see [4]).

Following the same notations as in the previous section, let us set an invertible sheaf $L_m = \mathcal{O}_X(m\Theta)$ on the principally polarized Abelian variety (X, Θ) of dimension g.

Let us assume that $k = mN$ and let L_k be the invertible sheaf $\mathcal{O}_X(k\Theta)$; on $X \times \cdot\cdot\stackrel{r}{\cdot} \times X = Z$ we consider the invertible sheaf

$$\mathcal{M}_N = p_1^* L_k \otimes \cdots \otimes P_N^* L_k \simeq p_1^* L_k^{\otimes N} \otimes \cdots \otimes P_N^* L_k^{\otimes N}.$$

Let us define the vector subspace $E_k(N) \subset H^0(Z, \mathcal{M}_N)$ by the following conditions:

$$s \in E_k(N) \Leftrightarrow \begin{cases} s \text{ is invariant with respect to the action of the } N\text{-torsion subgroup} \\ \quad \Delta(X_N) \subset Z \text{ and is odd with respect to the permutations} \\ \quad \text{acting on } H^0(Z, \mathcal{M}_N) = H^0(X, L_k) \otimes \cdots \otimes H^0(X, L_k). \end{cases}$$

Let us set $V_m = H^0(X, L_m)$ and $V_k = H^0(X, L_k)$. By the very definition, one has that

$$E_k(N) = \bigwedge^N V_k \cap \operatorname{Im} \varphi_N^* \subset V_k \otimes \stackrel{N}{\cdots} \otimes V_k,$$

where $\varphi_N^* : H^0(Z, \mathcal{R}_N) \to H^0(Z, \mathcal{M}_N) = V_k \otimes \cdot\cdot\stackrel{N}{\cdot} \otimes V_k$ is the addition homomorphism defined in Section 6.

Note that the factorization $\varphi_N^* = \pi_{1,\ldots,N} \circ \xi_N$ implies that

$$E_k^0(N) = \bigwedge^N V_k \cap \operatorname{Im} \xi_N^* \subseteq E_k(N) \subseteq V_k^{\otimes N}.$$

Let $E_i^{\pm} \subset H^0(Z, \mathcal{R}_N)$ be the subspaces of eigenvectors of the automorphism on $H^0(Z, \mathcal{R}_N)$ induced by $\sigma_i : X^N \to X^N$, $\sigma_i(x_1, \ldots, x_N) = (x_1, \ldots, -x_i, \ldots, x_N)$.

Proposition 3.1. *There exists a natural isomorphism*

$$E_k(N) \simeq \varphi_N^* H^0(Z, \mathcal{R}_N)_-.$$

$H^0(Z, \mathcal{R}_N)_-$ being the vector subspace of $H^0(Z, \mathcal{R}_N)$ defined as the intersection of the vector subspaces E_i^- with $i > 1$.

Proof. This is easy from the equality $E_k(N) = \bigwedge^N V_k \cap \operatorname{Im} \varphi_N^*$. □

We can give a more explicit description of the subspace $E_k^0(N)$.

From Remark 2.9, it follows that $\operatorname{Im} \xi_N^*$ is the vector subspace of $H^0(Z, \mathcal{R}_N)$ described in Proposition 2.7.

Let $V_m^{\pm}$ be the subspaces of eigenvectors of V_m with respect to the action of the involution $[-1]_X : X \to X([-1]_X(x) = -x)$. Then, we have the following proposition.

Proposition 3.2.

$$E_k^0(N) = \varphi_N^*(V_m \otimes V_m^- \otimes \overset{N-1}{\cdots} \otimes V_m^-).$$

Proof. One has only to observe that $\operatorname{Im} \xi_N^*$ is naturally identified with $V_m \otimes \cdots^N \otimes V_m$. □

In our interpretation of the FQHE, the vector subspace $E_k^0(N)$ is the space of wave functions of a system of N electrons.

4. Poincaré bundles and Fourier–Mukai transforms

Let (X, Θ) be a p.p.a.v. of dimension g and $\hat{X}$ its dual Abelian variety. Let $\mathcal{P}$ be a Poincaré bundle on $X \times \hat{X}$; $\mathcal{P}$ is the line bundle on $X \times \hat{X}$ given by the universal property of $\hat{X}$.

Given an invertible sheaf $L_m \simeq \mathcal{O}_X(m\Theta)$ on X (with $m > 0$), we can construct the invertible sheaf on $X \times \hat{X}$:

$$\mathcal{L}_m = \pi_X^* L_m \otimes \mathcal{P},$$

where $\pi_X : X \times \hat{X} \to X$ and $\pi_{\hat{X}} : X \times \hat{X} \to \hat{X}$ are the natural projections.

The Fourier–Mukai transform of L_m is (see [5,6] for details)

$$S(L_m) = \pi_{\hat{X}_*}(\pi_X^* L \otimes \mathcal{P}) = \pi_{\hat{X}_*} \mathcal{L}_m.$$

It is well known that $S(L_m)$ is a rank m^g vector bundle on $\hat{X}$.

We can interpret $\mathcal{L}_m$ as the family of line bundles over X, parameterized by $\hat{X}$, which are algebraically equivalent to L_m.

If we wish to generalize the results of Section 1 to the case of a “variable line bundle” L_m, we must perform the base change $X \times \hat{X} \to \hat{X}$ and replace L_m by $\mathcal{L}_m$.

We can then define on $Z \times \hat{X}$ the following line bundles:

$$\widetilde{\mathcal{M}_N} = \bar{M}^* \mathcal{L}_m \otimes \left(\bigotimes_{i<j} \bar{s}_{ij}^* \mathcal{L}_m \right), \qquad \widetilde{\mathcal{R}_N} = (\bar{p}_1^* \mathcal{L}_m \otimes \cdots \otimes \bar{p}_N^* \mathcal{L}_m) \otimes \left(\bigotimes_{\substack{i>j \\ j\geq 2}} \bar{s}_{ij}^* \mathcal{L}_m \right),$$

where $\bar{M}$ and $\bar{s}_{ij}$ are the morphisms $Z \times \hat{X} \to X \times \hat{X}$ defined by: $\bar{M} = M \times \mathrm{Id}_{\hat{X}}$, $\bar{s}_{ij} = s_{ij} \times \mathrm{Id}_{\hat{X}}$ and $\bar{p}_i : Z \times \hat{X} \to X \times \hat{X}$ are the natural projections.

Defining $\bar{\varphi}_N : Z \times \hat{X} \to Z \times \hat{X}$ as $\bar{\varphi}_N = \varphi_N \times \mathrm{Id}_{\hat{X}}$, we have that

$$\bar{\varphi}_N^* \widetilde{\mathcal{R}_N} \simeq \widetilde{\mathcal{M}_N},$$

and Corollary 2.2 implies that

$$\bar{\varphi}_N^* \widetilde{\mathcal{R}_N} \simeq \widetilde{\mathcal{M}_N} \simeq \bar{p}_1^* \mathcal{L}_m^{\otimes N} \otimes \cdots \otimes \bar{p}_N^* \mathcal{L}_m^{\otimes N} \otimes \pi_{\hat{X}}^* F$$

for some invertible sheaf F on $\hat{X}$.

Bearing in mind the applications to the FQHE, we are mainly interested in the bundles

$$W_N(L_m) = \pi_{\hat{X}_*}(M^* \mathcal{L}_m) = \pi_{\hat{X}_*}(\bar{M}^*(\pi_X^* L_m \otimes \mathcal{P})),$$

which describe the dynamics of the center-of-mass.

Our main result on the structure of $W_N(L_m)$ is as follows.

Theorem 4.1. *For every $N > 0$ and $m > 0$, $W_N(L_m)$ are vector bundles over $\hat{X}$ of rank m^g. These vector bundles are semi-stable with respect to the p.p. $\hat{\Theta}$ induced by Θ on $\hat{X}$. Moreover, for every $N \geq 2$, there exist natural isomorphisms $W_N(L_m) \tilde{\to} W_{N-1}(L_m)$.*

Proof. Proof of the existence of isomorphisms $W_N(L_m) \tilde{\to} W_{N-1}(L_m)$ is the same as the proof given in the case of elliptic curves.

Therefore, the proof of the theorem is reduced to the case of $W_1(L_m)$ which is precisely the Fourier–Mukai transform of L_m, which is well known to be a vector bundle of rank m^g (for $m > 0$).

We only have to prove the semi-stability of $W_1(L_m)$ with respect to $\hat{\Theta}$.

Let us compute the slope of $W_1(L_m)$: we consider the isogeny $\varphi_{L_m} : X \to \hat{X}$ of degree m^{2g} defined by

$$\varphi_{L_m} = T_x^* L_m \otimes L_m^{\otimes -1},$$

$T_x : X \to X$ being the translation by x. It is known [7] that

$$\varphi_{L_m}^* W_1(L_m) \simeq H^0(X, L_m) \otimes L_m^{\otimes -1}.$$

Let us set $\mathcal{O}_X(D) = \det W_1(L_m)$; one has

$$\varphi^*(D \cdot \hat{\Theta}^{g-1}) = \deg(\varphi) \deg(D) = m^{2g} \deg(D),$$

and

$$\varphi^*(D \cdot \hat{\Theta}^{g-1}) = \varphi^* D \cdot (\varphi^* \hat{\Theta})^{g-1} = (-m^{g+1}\Theta)(m^2\Theta)^{g-1} = -m^{3g-1} g_!.$$

Then, $\deg(D) = -m^{g-1} g_!$:

$$\mu(W_1(L_m)) = \frac{\deg W_1(L_m)}{\mathrm{rk}\, W_1(L_m)} = -\frac{g_!}{m}.$$

Let us recall that from the computations of Lange and Birkenhake [8] one easily deduces that given an invertible sheaf $\mathcal{M}$ on $\hat{X}$, one has that $c_1(\mathcal{M}) \cdot \hat{\Theta}^{g-1} = g_! \cdot c$ for some integer c. Thus, in the definition of semi-stability on $\hat{X}$, with respect to the polarization $\hat{\Theta}$, we can replace the degree $c_1(\mathcal{M}) \cdot \hat{\Theta}^{g-1}$, of an invertible sheaf $\mathcal{M}$, by the reduced degree

$$r \deg(\mathcal{M}) = \frac{c_1(\mathcal{M}) \cdot \hat{\Theta}^{g-1}}{g_!},$$

and the reduced slope

$$\mu_r(\mathcal{M}) = \frac{c_1(\mathcal{M}) \cdot \hat{\Theta}^{g-1}}{g_! \cdot \operatorname{rk} \mathcal{M}}.$$

Let $F \subseteq W_1(L_m)$ be a subbundle of rank $r < m^g$ and reduced degree $r \deg(F) = d$. One has to show that

$$\mu_r(F) = \frac{d}{r} \le \mu_r\,(W_1(L_m)) = -\frac{1}{m}.$$

But it is known that to prove the semi-stability condition for $W_1(L_m)$ it suffices to prove that it is satisfied by the subbundles of rk $= 1$; that is, we can assume that $r = 1$. In this case, the inequality is equivalent to $d < 0$.

Let us take the pullback of $F \subseteq W_1(L_m)$ with respect to the isogeny φ_{L_m}

$$\varphi_{L_m}^* F \subseteq \varphi_{L_m *} W_1(L_m) \simeq H^0(X, L_m) \otimes L_m^{\otimes -1},$$

and $r \deg \varphi_{L_m}^* F = m^{2g} d \le r \deg L_m^{\otimes -1} = -m < 0$. Then, one has that $d < 0$. □

5. Fractional quantum Hall states in multi-layer two-dimensional electron systems

For applications to the FQH effect, we shall apply the theory developed in previous sections to the following situation.

Let us consider the forms $E = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ defined by $\tau \in \mathbb{H}_1$ (upper half-plane) and let us denote by $e \in E$ the origin of the group law of E. The natural polarization on E is given by the invertible sheaf $\mathcal{O}_E(e)$.

For any positive integer $g \in \mathbb{Z}$, let us denote by X_g the Abelian variety

$$X_g = E \times \overset{g}{\cdots} \times E.$$

Let $X_g \to^{q_i} E$ be the natural projection into the ith factor. One can define a p.p., Θ, on X_g as follows:

$$\mathcal{O}_X(\Theta) = \bigotimes_{i=1}^{g} q_i^* \mathcal{O}_E(e).$$

Let K be a symmetric, positive, integer-valued $g \times g$ matrix. This matrix defines an isogeny

$$K : E^g = X \to E^g = X.$$

One can define a line bundle L_k on X by

$$L_k = K^* \mathcal{O}_X(\Theta).$$

We can apply the results of Sections 1 and 2 to this sheaf.

Let $N > 0$ be an integer number, $r = (N(N-1)/2) + 1$, and ξ_N, φ_N the morphisms defined in Section 1

$$\xi_N : X \times \overset{N}{\cdots} \times X \simeq E^{gN} \to X \times \overset{r}{\cdots} \times X, \qquad \varphi_N : X \times \overset{N}{\cdots} \times X \to X \times \overset{N}{\cdots} \times X.$$

On $Z = X^N$, one has the sheaf

$$\mathcal{R}_N = (p_1^* L_K \otimes \cdots \otimes p_N^* L_K) \otimes \left(\bigotimes_{\substack{i>j \\ j \geq 2}} s_{ij}^* L_K \right),$$

and isomorphisms

$$\xi_N^* \left(\bigotimes_{i=1}^{r} p_i^* L_K \right) \simeq \varphi_N^* R_N \simeq \bigotimes_{i=1}^{N} p_i^* L_K^{\otimes N} \simeq \mathcal{M}_N(K).$$

Analogous to Section 2, for each matrix K we can define the vector subspace $E_K(N) \subset H^0(Z, \mathcal{M}_N(K))$ which will be identified with the Hilbert space of our problem

$$s \in E_K(N) \Leftrightarrow \begin{cases} s \text{ is invariant with respect to the action of the subgroup} \\ \quad \Delta(X_N) \subset Z \text{ and is odd with respect to the permutations} \\ \quad \text{acting on } H^0(Z, \mathcal{M}_N(K)) = H^0(X, L_K^{\otimes N}) \otimes \cdots \otimes H^0(X, L_K^{\otimes N}). \end{cases}$$

Also one has that

$$E_K(N) = \bigwedge^N H^0(X, L_K^{\otimes N}) \cap \operatorname{Im} \varphi_N^*.$$

Analogous to Section 2 we can also define the subspace $E_K^0(N) = \bigwedge^N H^0(X, L_K^{\otimes N}) \cap \operatorname{Im} \xi_k^*$.

Let us denote a point of X^N by $(x_1, \ldots, x_N)$ and $x_i = (t_1^i, \ldots, t_g^i) \in E^g = X$.

The explicit computations can be performed along the lines of Mumford [2,3].

Note that the kernel of the isogeny $K : X \to X$ can be identified with the finite subgroup

$$X_K \simeq \mathbb{Z}^g / K\mathbb{Z}^g \times \mathbb{Z}^g / \hat{K}\mathbb{Z}^g.$$

The order of this group is $|X_K| = |\det K|^2$ and $H^0(X, L_K)$ is a $\mathbb{C}$-vector space of dimension $|\det K|$. Obviously, one has

$$K(L_K) = X_K \subset K(L_K^{\otimes N}), \qquad N \cdot K(L_K^{\otimes N}) = K(L_K).$$

Let us set $V = H^0(X, L_K)$ and $V_K = H^0(X, L_K^{\otimes N})$. One has the analogous results of those proved in Sections 2 and 3 and

$$E_K(N) = \varphi_N^* H^0(Z, \mathcal{R}_N)_-, \qquad E_K^0(N) = \varphi_N^* (V \otimes V^- \otimes \overset{N-1}{\cdots} \otimes V_-).$$

Moreover, given $d = (d_1, \ldots, d_N) \in N \cdot B(L_K)^N = [\mathbb{Z}^g / K\mathbb{Z}^g]^N \subseteq B(\mathcal{R}_N)$, let us denote by δ_d the element

$$\delta_d = \delta_{d_1} \otimes \cdots \otimes \delta_{d_N} \otimes \left(\bigotimes_{\substack{i>j \\ j \geq 2}} s_{ij}^* \delta_{d_i - d_j} \right) \in H^0(Z, \mathcal{R}_N).$$

It follows that the vector subspace $E_K^0(N)$ is generated by the sections $\varphi_N^*(\delta_d)$ and one has the identity

$$\begin{aligned}\varphi_N^*(\delta_d) &= \theta[d_1](x_1+\cdots+x_N)\prod_{j\geq 2}\theta[d_j](x_1-x_j)\prod_{\substack{i>j\\ j\geq 2}}\theta[d_i-d_j](x_i-x_j)\\ &= \lambda \sum_{\substack{b_1+\cdots+b_N=d_1\\ b_1-b_2=d_2\\ \vdots\\ b_1-b_N=d_N}} \theta[b_1](x_1)\theta[b_2](x_2)\cdots\theta[b_N](x_N),\end{aligned} \tag{1}$$

where $(x_1,\ldots,x_N)\in X\times\cdots^N\times X = E^{gN}$, that is, $x_i=(z_{i1},\ldots,z_{ig})\in E^g$.

Observe that

$$\mathbb{Z}^g/K\mathbb{Z}^g \simeq \mathbb{Z}/n_1\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/n_g\mathbb{Z}$$

for some integers $n_1,\ldots,n_g$ such that $\det K = n_1,\ldots,n_g$.

Then, in the above statements $d_1,\ldots,d_N$ are elements of the group $\mathbb{Z}/n_1\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/n_g\mathbb{Z}$ (once one has fixed the corresponding theta-structures).

6. Filling factors and Hall conductivity

In a multi-layer many-electron system where the fractional quantum Hall effect is observed, the ground state is a quantum fluid with several possible topological orders; see [9]. The different phases are characterized by the $g\times g$ matrix

$$K=\begin{pmatrix} 2p+1 & 2p & \cdots & 2p\\ 2p & 2p+1 & \cdots & 2p\\ \vdots & \vdots & \ddots & \vdots\\ 2p & 2p & \cdots & 2p+1\end{pmatrix},$$

where p is an integer greater than zero and g the number of layers.

The ground state wave function

$$\tilde{\psi} = \prod_{\substack{i,j=1\\ i<j}}^{N}\left[\prod_{a=1}^{g}(z_i^a-z_j^a)^{2p+1}\prod_{a<b}(z_i^a-z_j^b)^{2p}\right]\exp\left[-\sum_{a=1}^{g}\sum_{i=1}^{N}|z_i^a|^2\right]$$

is the generalization of the Laughlin state to the case in which each layer is isomorphic to $\mathbb{C}$; here, z_i^a is the ith particle position in the ath layer, and we assume that there are N particles per layer, so that the total number of particles is $N_T=gN$.

We focus on this problem when each electron moves on a torus; the one-particle configuration space is the elliptic curve $E=\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ of the previous sections. The modular parameter $L_2\,e^{i\theta}/L_1$ encodes the periodicities of the basic lattice, which is the same for every layer. A constant magnetic field B allows for a well-behaved quantum system, compatible with the lattice and the order “meant” by the matrix K, if and only if

$$K_1 = \begin{pmatrix} (2p+1)N & 2pN & \cdots & 2pN \\ 2p & 2p+1 & \cdots & 2p \\ \vdots & \vdots & \ddots & \vdots \\ 2p & 2p & \cdots & 2p+1 \end{pmatrix}, \qquad \frac{eB}{\hbar c} L_1^2 = \frac{2\pi |\det K_1|}{\operatorname{Im}\tau}.$$

Here, e, $\hbar$ and c are, respectively, the electron charge, the Planck constant and the speed of light in vacuum. The quantum space of one-particle states is the space of sections of the line bundle $L_{K_1} = K_1^* \theta_{X_g}(\Theta)$ and the first Landau level corresponds to the subspace of holomorphic sections $H^0(X_g, L_{K_1})$.

There is a many-electron wave function proposed by Haldane and Rezayi [10,11] as the ground state for the quantum Hall fluid in a periodic lattice. Both the HR wave function and its generalization to a multi-layer are of Laughlin type and the framework for the mathematical understanding of such complex quantum states is provided by the developments set forth before in this paper. We start by noticing that the isomorphism established at the end of Section 4 now reads

$$\mathbb{Z}^g / K\mathbb{Z}^g \simeq \mathbb{Z}/(2gp+1)\mathbb{Z} \oplus 1 \oplus 1 \oplus \cdots \oplus 1,$$

i.e. $n_1 = (2gp+1), n_2 = n_3 = \cdots = n_g = 1$ because these are the eigenvalues of the K matrix.

The center-of-mass dynamics and the relative motion of each pair of particles produce contributions that factorize in the ground state wave function. In a basis in X_g in which K is diagonal:

1. The center-of-mass wave function is a theta function of g variables that we write following the conventions of Ref. [12] in order to translate the developments of the previous sections to the notation used in the physics literature:

$$F_{CM}(\vec{X}) = \Theta \begin{bmatrix} d_1 K_D^{-1} \vec{e}_1 \\ \vec{0} \end{bmatrix} (K_D \vec{X} | K_D \tau), \qquad \vec{X} = \vec{x_1} + \vec{x_2} + \cdots + \vec{x_N},$$

   $\vec{X}$ is the CM coordinate, K_D is a diagonal matrix such that $\det K_D = \det K$ (we have chosen $K_{D_{11}} = 2gp+1$) and the vector of g components $\vec{e}_1$ is $(1, 0, \ldots, 0)$.

   This expression for the center-of-mass wave function is exactly the same as $\theta[d_1](x_1 + x_2 + \cdots + x_N)$ in the previous section and, undoing the diagonalization, one obtains

$$F_{CM}(\vec{Z}) = \Theta \begin{bmatrix} K^{-1} \vec{\alpha} \\ \vec{0} \end{bmatrix} (K\vec{Z} | K\tau),$$

   where $\vec{Z} = \vec{z_1} + \vec{z_2} + \cdots + \vec{z_N}$ is the CM coordinate in a basis of X_g, where K is not diagonal, and $\vec{\alpha} \in \mathbb{Z}^g / K\mathbb{Z}^g$. This is the form in which it appears in the physics literature.

2. The factor in the ground state wave function due to relative motion has the form: if $\vec{x}_{ij} = \vec{x}_i - \vec{x}_j$,

$$F_r(\vec{x}_{ij}) = \prod_{i<j} \Theta_- \begin{bmatrix} d_{ij}^- K_D^{-1}\vec{e}_1 \\ \vec{0} \end{bmatrix} (K_D\vec{x}_{ij}|K_D\tau),$$

$$d_{ij}^- = d_i^- - d_j^-, \quad i \geq 2, \quad d_{ij} = d_j^-, \quad i = 1, \quad d_{ij}^- = 1, 2, \ldots, gp.$$

Fermi statistics requires the use of anti-symmetric functions in $\vec{x}_{ij} \mapsto -\vec{x}_{ij}$

$$\Theta_- \begin{bmatrix} d_{ij}^- K_D^{-1}\vec{e}_1 \\ \vec{0} \end{bmatrix} (K_D\vec{x}_{ij}|K_D\tau)$$
$$= \frac{1}{2}\left(\Theta \begin{bmatrix} d_{ij}^- K_D^{-1}\vec{e}_1 \\ \vec{0} \end{bmatrix} (K_D\vec{x}_{ij}|K_D\tau) - \Theta \begin{bmatrix} -d_{ij}^- K_D^{-1}\vec{e}_1 \\ \vec{0} \end{bmatrix} (K_D\vec{x}_{ij}|K_D\tau) \right).$$

Nevertheless, the ground state wave function

$$\psi = F_{CM}(\vec{X}) F_r(\vec{x}_{ij}) \exp\left\{ -\frac{1}{4} \sum_i (\mathrm{Im}\,\vec{x}_i)(\mathrm{Im}\,\vec{x}_i) \right\},$$

apart from the non-analytic exponential factor, consists of terms of the form of the left-hand member of formula (1).

Therefore, ψ can also be expressed as a product of theta functions in the $\vec{x}_i$ variables with characteristics $b_i \in \mathbb{Z}/(2gp+1)\mathbb{Z} \oplus 1 \oplus 1 \oplus \cdots \oplus 1$ related to the K_D matrix.

In the physics of the quantum Hall effect, the concept of the filling factor plays a central role; if the magnetic field is strong enough to provide more states in the first Landau level than electrons, it is defined as

$$f = \frac{\text{number of particles}}{\text{number of states in the first LL}},$$

and the Hall conductivity is studied as a function of f.

If the number of states in the first LL is a finite number, $\dim H^0(X_g, L_{K_1}) = \det(K_1)$ in our case, then f is

$$f_{\mathrm{HR}} = \frac{N_T}{\det K_1} = \frac{g}{2gp+1}.$$

Different integers g, and hence different values of f_{HR}, give rise to a hierarchy of experimentally observed topological orders: associated with each f of this form there are quantum fluids that arise as ground states of the fractional quantum Hall effect without periodic boundary conditions.

What we have shown by proving the generalized addition formulae for Abelian varieties is that the fractional quantum Hall states in multi-layer two-dimensional electron systems are compatible with periodic lattices. Only the existence of such addition formulae makes it possible to claim that the generalized Haldane–Rezayi wave function implies $f_{\mathrm{HR}} = g/(2gp+1)$.

In fact, a further development remains to be made in order to make contact with the HR ground state. We remark that there is a linear combination such that

$$\sum_{d_{ij}^-=1}^{gp} c[d_{ij}^-]\Theta_- \begin{bmatrix} d_{ij}^- K_D^{-1}\vec{e}_1 \\ \vec{0} \end{bmatrix} (K_D\vec{X}_{ij}|K_D\tau) = \Theta^{2gp+1}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(x_i^1 - x_j^1|\tau)\prod_{a=2}^{g}\Theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(x_i^a - x_j^a|\tau),$$

appearing in the right-hand member odd theta functions of one variable. Undoing the diagonalization of K, one easily checks that

$$\prod_{a=1}^{g}\Theta^{2p+1}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z_i^a - z_j^a|\tau)\prod_{a<b}\Theta^{2p}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z_i^a - z_j^a|\tau) \simeq \Theta^{2gp+1}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(x_i^1 - x_j^1|\tau)\prod_{a=2}^{g}\Theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(x_i^a - x_j^a|\tau)$$

in such a way that the HR wave function can be traced back to the above ψ.

The generalized addition formulae are valid for any Abelian variety $X_g = \mathcal{C}^g/(\mathcal{Z}^g \oplus \Omega\mathcal{Z}^g)$, with Ω a matrix in the Siegel upper half-space of rank g in $\mathbb{H}_g$. In the application to the quantum Hall effect, we have restricted ourselves to the case $X_g = E^g$, i.e. $\Omega = \tau I_{g\times g}$. There is no difficulty in extending the analysis to any $\Omega \in \mathbb{H}_g$ that physically corresponds to taking into account different periodicities for different layers and a tunnel effect of weak amplitude between layers, a situation also considered by condensed matter physicists, see [13]. It is also convenient to make a brief comment on the second type of addition formulas; Proposition 2.7, from a physical point of view. Mathematically, the origin of such addition formulas is the freedom of choosing the isogeny $\varphi_N : Z \to Z$: there are different projections from $X_g \times \cdots^r \times X_g$ to $Z = X_g \times \cdots^N \times X_g$ ($r = [N(N-1)/2] + 1$). Another choice of $\pi_{1,\dots,N}$, for instance, would lead one to define

$$\varphi_N'(x_1,\dots,x_N) = (x_1 + x_2 + \cdots + x_N, x_2 - x_1, x_2 - x_3, \dots, x_2 - x_N),$$

i.e. it would singularize relative coordinates with respect to the second particle. In quantum mechanics particles are indistinguishable and thus this possibility is physically equivalent to choosing φ_N based on the first particle coordinate. For this reason the wave functions invariant under the second subgroup of $K(\mathcal{R}_N)$ do not enter in physical arguments, and the ordering $i < j$ is chosen as the most natural one.

Further knowledge of the implications of the nature of the HR ground state wave function can be obtained by means of a gedanken experiment, see [14]: magnetic fluxes are induced by two solenoids per layer connected to the Hall device in such a way that they are compatible with the electrons if

$$\mathrm{Re}\,\vec{\phi} \in \left[\vec{0}, \frac{\hbar c}{e}\vec{u}\right], \qquad \mathrm{Im}\,\vec{\phi} \in \left[\vec{0}, \frac{\hbar c}{e}\vec{u}\right],$$

according to the Aharanov–Bohm effect. Here, $\vec{\phi}$ is a complex g vector which encodes the solenoid fluxes and $\vec{u} = (1, 1, \ldots, 1)$ is a real constant g vector. The generalized HR states are modified to

$$\psi_{d_1}[\vec{\phi}] = F_{CM}^{d_1}[\vec{\phi}; \vec{X}] F_r[\vec{X}_{ij}] \exp\left\{-\frac{1}{4}\sum_i [(\mathrm{Im}\,\vec{X}_i)^{\mathrm{t}}\,\mathrm{Im}\,\vec{X}_i]\right\},$$

$$F_{CM}^{d_1}[\vec{\phi}; \vec{X}] = \Theta\begin{bmatrix} d_1 K_D^{-1}\vec{e}_1 + \vec{\phi}_1 \\ \vec{\phi}_2 \end{bmatrix}(K_D\vec{X}_{ij}|K_D\tau),$$

where $\vec{\phi}_1 = (e/\hbar c)\,\mathrm{Re}\,\vec{\phi}$ and $\vec{\phi}_2 = (e/\hbar c)\,\mathrm{Im}\,\vec{\phi}$. The relative motion is not affected but the contribution of the center-of-mass dynamics to the ground state is modified by including the solenoid fluxes as characteristics of the theta function.

Mathematically, one must interpret $\vec{\phi}$ as points in the Jacobian $\hat{X}_g$ of X_g and we proceed to identify the bundle where $\psi^{d_1}[\vec{\phi}]$ is defined as a section, using the developments of Section 3. In fact, only the replacement of L_m by L_K is necessary. We thus start by constructing the invertible sheaf

$$\mathcal{L}_K = \pi_X^* L_K \otimes \mathcal{P},$$

a family of line bundles over X parameterized by $\hat{X}$, and defining the Fourier–Mukai transform of L_K

$$S(L_K) = \pi_{\hat{X}_*}(\pi_X^* L_K \otimes \mathcal{P}) = \pi_{\hat{X}_*}\mathcal{L}_K,$$

$S(L_K)$ is a vector bundle over $\hat{X}$ of rank $(\det K)^g$ whose fibers are vector spaces of dimension $(\det K)^g$ whose bases are provided by the basis of $H^0(\hat{X}, L_K)|_{\hat{x}_0 \in \hat{X}}$. Taking this into account, one easily recognizes that

$$s^{d_1} = F_{CM}^{d_1}[\vec{\psi}; \vec{X}] F_r[\vec{x}_{ij}]$$

is a holomorphic section in the bundle

$$\tilde{\mathcal{M}}_N^K = \bar{M}^* \mathcal{L}_K \otimes \left(\bigotimes_{i<j} \bar{s}_{ij}^* \mathcal{L}_K\right)$$

defined in perfect analogy with the bundle $\tilde{\mathcal{M}}_N$ of Section 3: one merely replaces $\mathcal{L}_m$ by $\mathcal{L}_K$.

We now focus on the center-of-mass dynamics. Taking direct image amounts to integrate over the variables in the other factors and we find

$$S_{CM}^{d_1} = F_{CM}^{d_1}[\vec{\phi}] = \int_X \mathrm{d}\,\mathrm{vol}_X\, F_{CM}^{d_1}[\vec{\phi}; \vec{X}],$$

which determines the contribution of the solenoid fluxes to the CM ground state wave function; this is a holomorphic section in the Fourier–Mukai transform of the bundle $\bar{M}^* L_K$

$$W_N(L_K) = \pi_{\hat{X}_*}(\bar{M}^*\mathcal{L}_K) = \pi_{\hat{X}_*}(\bar{M}^*(\pi_X^* L_K \otimes \mathcal{P})).$$

From Section 3 we know that $W_N(L_K) \simeq W_{N-1}(L_K)$ and the slope and reduced slope of $W_1(L_K)$ are given by

$$\mu(W_1(L_K)) = -\frac{g(\det K)^{g-1} g_!}{(\det K)^g} = -\frac{g g_!}{\det K}, \qquad \mu_r(W_1(L_K)) = -\frac{g}{\det K}.$$

There is a novelty: the factor g appears due to the freedom of choosing $2gp+1$ as any of the g eigenvalues of K.

The Hall conductivity of the system is expressed in perturbation theory by the Kubo–Thouless formula [12]

$$\sigma_{\mathrm{H}} = \frac{\mathrm{i}}{2\pi}\frac{ge^2}{r\hbar}\sum_{d_1=1}^{r}[\langle\vec{\nabla}_1\psi^{d_1}|\vec{\nabla}_2\psi^{d_1}\rangle - \langle\vec{\nabla}_2\psi^{d_1}|\vec{\nabla}_1\psi^{d_1}\rangle],$$

where $r = \det K$, $\vec{\nabla}_a = \partial/\partial\vec{\phi}_a$ and $\langle|\rangle$ defines the L^2-norm

$$\langle f|g\rangle = \int_{X^{\otimes N}} \mathrm{d}\,\mathrm{vol}_{X^{\otimes N}}\, f^*(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N) g(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N).$$

This formula can be interpreted as follows: from the section ψ^{d_1}, we obtain a connection, for any d_1,

$$\omega^{d_1} = -2\,\mathrm{Im}\langle\psi^{d_1}|\vec{\nabla}_2\psi^{d_1}\rangle\,\mathrm{d}\vec{\phi}_2$$

in a certain line bundle over $\hat{X}$. The curvature

$$\mathcal{R}_{\omega^{d_1}} = 2\pi\,\mathrm{d}\vec{\phi}_1 \wedge \mathrm{d}\vec{\phi}_2$$

is constant on $\hat{X}$ and therefore σ_{H} is equal to its average value $\langle\sigma_{\mathrm{H}}\rangle$

$$\langle\sigma_{\mathrm{H}}\rangle = \frac{\mathrm{i}}{2\pi}\frac{ge^2}{r\hbar}\sum_{d_1=1}^{r}[\langle\vec{\nabla}_1\psi^{d_1}|\vec{\nabla}_2\psi^{d_1}\rangle - \langle\vec{\nabla}_2\psi^{d_1}|\vec{\nabla}_1\psi^{d_1}\rangle] = \frac{g}{2gp+1}.$$

The bundle is therefore $W_1(L_K)$ and the Hall conductivity is a topological invariant, the reduced slope of $W_1(L_N)$

$$\sigma_{\mathrm{H}} = |\mu_r(W_1(L_K))|.$$

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